

Poincaré Covariance and κ -Minkowski Spacetime

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Abstract

A fully Poincaré covariant model is constructed as an extension of the κ -Minkowski spacetime. Covariance is implemented by a unitary representation of the Poincaré group, and thus complies with the original Wigner approach to quantum symmetries. This provides yet another example (besides the DFR model), where Poincaré covariance is realised à la Wigner in the presence of two characteristic dimensionful parameters: the light speed and the Planck length. In other words, a Doubly Special Relativity (DSR) framework may well be realised without deforming the meaning of “Poincaré covariance”.

1 Introduction

According to DSR scenario [1, 2], to account for phenomena at Planck scale (possibly in view of a theory of quantum gravity), ordinary Poincaré covariance should be properly modified, so to incorporate two universal dimensionful parameters: the light speed $c \sim 3 \times 10^8 \text{ cm} \cdot \text{s}^{-2}$ and the Planck length $\lambda_P \sim 10^{-33} \text{ cm}$. It has been proposed [3] to test such a scenario on the so-called κ -Minkowski spacetime, where the name refers to the notation $\kappa = 1/\lambda_P$ [4, 5]. In that model, Poincaré covariance is deformed in the sense of quantum groups.

While equations which are not form-invariant may well be compatible with the principle of equivalence of Lorentz observers (e.g. the Coulomb gauge), a deformation of the notion of covariance at Planck scale would allow to select a privileged class of observers.

Moreover, the Doplicher-Fredenhagen-Roberts (DFR) model [6, 7], proposed in 1994, shows that the usual transformation laws of special relativity may well be “doubly special”, without any deformation. In fact, therein the commutation relations with the corresponding quantisation of ordinary functions depend on λ_P , and they are covariant under the adjoint action of unitary operators representing the ordinary (undeformed) Poincaré group. This contrasts the

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widespread folklore that the presence of two universal dimensionful parameters should necessarily force us into the realm of “modified” Poincaré covariance.

In this letter we provide another example of a model with Poincaré covariance à la Wigner, and two dimensionful parameters ($c = \kappa = 1$, in natural units). The (non trivial) defining relations are

$$[X^\mu, X^\nu] = i(V^\mu(X - A)^\nu - V^\nu(X - A)^\mu), \quad (1a)$$

$$V_\mu V^\mu = I, \quad (1b)$$

where A^μ, V^μ are central, and the usual Lorentz metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is used. Commutation relations are understood to hold strongly; in particular the regular (Weyl) form of (1a) is

$$e^{ihX} e^{ikX} = e^{i(h+k-\varphi(h,k;V))_\mu A^\mu} e^{i\varphi_\mu(V;h,k)X^\mu}, \quad (2)$$

where φ is given by equations (7).

More precisely, we claim that it is possible to construct the selfadjoint operators X^μ, V^μ, A^μ together with a unitary representation U of the full Poincaré group \mathcal{P} , such that

$$U(\Lambda, a)^{-1} X^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu X^\nu + a^\mu I, \quad (3a)$$

$$U(\Lambda, a)^{-1} V^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu V^\nu, \quad (3b)$$

$$U(\Lambda, a)^{-1} A^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu A^\nu + a^\mu I \quad (3c)$$

This is obtained by “full covariantisation” of the κ -Minkowski model. The intermediate step of Lorentz covariantisation was discussed in [8], in a more general setting. The covariantised model contains the initial model as a component; in a sense (made precise in [8]) it is the smallest possible covariant central extension of the usual κ -Minkowski model. Note that, analogously, the DFR model could be equivalently described as the minimal central full covariantisation of a “canonical quantum spacetime”. The procedure is inspired by the mathematical construction of crossed products (also known as covariance algebras).

We take the point of view of the DFR analysis, where only the commutation relations among the coordinates (and their commutators) have to be modified; momenta (defined as the generators of translations) forcefully commute pairwise, while their commutation relations with the coordinates simply express the action of infinitesimal translations of the coordinates, as usual. In other words, we quantise the configuration space, which amounts to find a candidate for replacing the localisation algebra of quantum field theory. We do not quantise the phase space, namely we do not aim to a “more non-commutative” version of quantum mechanics (see [7] for a general discussion about motivations, and [9] for a review). Quantum field theoretical aspects will be discussed elsewhere.

2 The Covariant Coordinates

Covariant representations of the relations (1) will necessarily be highly reducible; let us first take for X^μ, V^μ, A^μ an irreducible regular representation. By the

Schur's lemma, $V^\mu = v^\mu I, A^\mu = a^\mu I$ for some real 4-vectors v, a , so that

$$[X^\mu, X^\nu] = i(v^\mu(X^\nu - a^\nu I) - v^\nu(X^\mu - a^\mu I)) \quad (4)$$

and $v_\mu v^\mu = 1$. We recognise as a special case the usual κ -Minkowski relations, corresponding to the choice $v = v_{(0)}, a = 0$, where

$$v_{(0)} = (1, 0, 0, 0).$$

For this particular case, the irreducible representations are all known [8]: $X_{(0)}^\mu$ will denote the corresponding universal representation (containing all irreducibles precisely once), see the appendix. The operators $X_{(0)}^\mu$ act on the Hilbert space $\mathfrak{H}_{(0)}$, with scalar product $(\cdot, \cdot)_{(0)}$.

Now, for every $(\Lambda, a) \in \mathcal{P}$, the operators

$$X^\mu = (\Lambda X_{(0)} + aI)^\mu = \Lambda^\mu{}_\nu X_{(0)}^\nu + a^\mu I$$

fulfill (4) with $v = \Lambda v_{(0)}$. In this way, it is possible to obtain representation for any pair $(v, a) \in H \times \mathbb{R}^4$, where $H = \mathcal{L}v_{(0)}$ is the orbit of $v_{(0)}$ under the full Lorentz group $\mathcal{L} = O(1, 3)$; H is a two sheeted hyperboloid.

By taking a direct integral over the Haar measure $d(\Lambda, a)$ of \mathcal{P} of all the irreducible representations so constructed, it is easy (see e.g. the analogous discussion of [7, 8]) to construct selfadjoint operators X^μ, V^μ, A^μ and a unitary representation U of \mathcal{P} , fulfilling (1,3). The result of the construction is equivalent to the following covariant representation. Consider the Hilbert space of $\mathfrak{H}_{(0)}$ -valued functions $\psi(\Lambda, a)$, with scalar product

$$(\psi, \psi') = \int_{\mathcal{P}} d(\Lambda, a) (\psi(\Lambda, a), \psi'(\Lambda, a))_{(0)}.$$

Then set

$$(X^\mu \psi)(\Lambda, a) = (\Lambda X_{(0)} + aI)^\mu \psi(\Lambda, a), \quad (5a)$$

$$(V^\mu \psi)(\Lambda, a) = (\Lambda v_{(0)})^\mu \psi(\Lambda, a), \quad (5b)$$

$$A^\mu \psi(\Lambda, a) = a^\mu \psi(\Lambda, a), \quad (5c)$$

$$U(M, b)\psi(\Lambda, a) = \psi((M, b)^{-1}(\Lambda, a)). \quad (5d)$$

Above (Λ, a) and (M, b) are elements of \mathcal{P} . Note that U is a strongly continuous representation of \mathcal{P} , and we may define momentum operators P^μ by setting $e^{ia^\mu P_\mu} = U(\mathbb{I}, a)$; they fulfil the commutation relations

$$[P^\mu, P^\nu] = 0, \quad [P^\mu, V^\nu] = 0, \quad (6a)$$

$$[P^\mu, A^\nu] = [P^\mu, X^\nu] = ig^{\mu\nu} I. \quad (6b)$$

Clearly, the first equation of (6a) follows from the abelianness of the translation subgroup; the second expresses translation invariance of V , while (6b) describes the effect of infinitesimal translations on X, A .

Analogously the generators of infinitesimal Lorentz transformations describe the usual (undeformed) action of the infinitesimal Lorentz transformations on $X^\mu, V^\mu, A^\mu, P^\mu$.

Note also that the orthogonal projection $E\psi = \chi\psi$, where χ is the characteristic function of the Lorentz subgroup \mathcal{L} of \mathcal{P} , fulfills $[E, X^\mu] = 0$; it reduces the model to the Lorentz covariant component discussed in [8]. Indeed, we have $[E, U((A, 0))] = 0$ always, even though E does not commute with $U(A, a)$ for $a \neq 0$.

We remark that, at a formal level, the right hand side of (4) may be regarded as a combination of a “canonical” and a “Lie type” contribution, according to a popular terminology.

3 Weyl Symbols and Relations

In order to define a star product, we consider functions $f = f(v, a; x)$, where x runs in the classical Minkowski spacetime \mathbb{R}^4 , while (v, a) runs in the parameter space $H \times \mathbb{R}^4$ (surviving the classical limit as extra dimensions). Then the quantisation prescription is

$$f(V, A; X) = \frac{1}{(2\pi)^2} \int dk \hat{f}(V, A; k) e^{ikX}.$$

Above, the replacement of the variables v^μ, a^μ by the operators V^μ, A^μ respectively is understood in the usual sense of functions of pairwise commuting operators. The replacement of x by X mimics instead the usual Weyl prescription, where

$$\hat{f}(V, A; k) = \frac{1}{(2\pi)^2} \int dx f(V, A; x) e^{-ikx}.$$

One may obtain the symbolic calculus with the star product defined by

$$f(V, A; X)g(V, A; X) = (f \star g)(V, A; X).$$

Quantisation intertwines operator adjunction and pointwise conjugation:

$$\bar{f}(V, A; X) = f(V, A; X)^*.$$

In order to carry over explicit computations, we need to know the Weyl relations (2) explicitly, namely to compute φ . To this end, we first consider the irreducible case, where $V = vI, A = aI$. We use the standard space-time notations $v = (v^\mu) = (v^0, \vec{v})$, $(v_\mu) = (v^0, -\vec{v})$, where $\vec{v} \in \mathbb{R}^3$; the metric has signature $(+ - - -)$.

We now think of $\vec{v}, \vec{h}, \vec{k}, \vec{0}$ as row 3-vectors; $\vec{v}^t, \vec{h}^t, \vec{k}^t, \vec{0}^t$ are the corresponding column vectors (where t indicates row-by-column transposition). The Lorentz matrix

$$A = \left(\begin{array}{c|c} v^0 & \vec{v} \\ \hline \vec{v}^t & \mathbb{I} + \frac{\vec{v}^t \vec{v}}{1+v^0} \end{array} \right)$$

fulfils $\Lambda v^{(0)} = v$. We now want to compute the Weyl relations for the operators $X = \Lambda X_{(0)} + aI$. We observe that by definition, $e^{ikX} = e^{ika} e^{i(\Lambda^{-1}k)X_{(0)}}$, hence by (8,9) we have

$$\begin{aligned} e^{ihX} e^{ikX} &= e^{i(h+k)a} e^{i\phi(\Lambda^{-1}h, \Lambda^{-1}k)X_{(0)}} \\ &= e^{i(h+k-\varphi(h,k;v))a} e^{i\varphi(h,k;v)X}, \end{aligned}$$

where

$$\varphi(h, k; v) = \Lambda\phi(\Lambda^{-1}h, \Lambda^{-1}k).$$

A routine computation yields

$$\begin{aligned} \varphi^0(h, k; v) &= w((h+k)v, hv) e^{ikv} (v^0 \vec{h} - h^0 \vec{v}) \vec{v}^T + \\ &\quad + w((h+k)v, kv) (v^0 \vec{k} - k^0 \vec{v}) \vec{v}^T + \\ &\quad + (h+k)v v^0, \end{aligned} \tag{7a}$$

$$\begin{aligned} \vec{\varphi}(h, k; v) &= w((h+k)v, hv) e^{ikv} (hv \vec{v} - \vec{h}) + \\ &\quad + w((h+k)v, kv) (kv \vec{v} - \vec{k}v) + \\ &\quad - (h+k)v \vec{v}. \end{aligned} \tag{7b}$$

Note that, as expected, φ does not depend on the particular choice of Λ , provided that $\Lambda v_{(0)} = v$.

To finally obtain (2) for the most general regular representation, it is sufficient to replace the variables v, a with the operators V, A , in the usual sense of functions of pairwise commuting operators.

4 Conclusions

A universal length may well exist in a Poincaré covariant setting. There is no contradiction between this statement and the Lorentz-Fitzgerald contraction. Indeed, the Planck length plays the rôle of a *characteristic* length ruling the structure of commutation relations; it is not an observable quantity itself. It follows that the quest for a universal length *alone* does not provide a motivation for deforming the notion of covariance.

Note that the approach to covariantisation followed here essentially consists in extending the algebra of coordinates by a centre which provides the room where to accommodate the joint spectrum of the additional operators. This centre is already classical, and survives the large scale limit as a manifold of extra dimensions. See also [10].

It should be mentioned that this covariantisation method does not exploit the symmetries which are possibly already present in the basic commutation relations. In particular, when applied to the κ -Minkowski, it does not extend covariance under the natural action of rotations on the initial set of generators. Covariance is instead implemented by an action of rotations (as elements of the Poincaré group) which replaces the original one.

By similar methods it is possible to covariantise the lightlike ($V_\mu V^\mu = 0$) and spacelike ($V_\mu V^\mu = -I$) models proposed in [11]. Indeed, this method is quite general and can be applied to a large class of models with Lie relations, provided that they have a good representation theory.

This however does not exhaust the possibilities. For example, there is a variant of the DFR model where the commutators of the Lorentz covariant coordinates are not central [12, 13].

The model proposed here allows for discussing general features of coordinate quantisation. It should be stressed however that no direct physical motivations has been devised for this particular choice of commutation relations. Although the lack of covariance has been cured by covariantisation, one of our main criticisms to the κ -Minkowski model still survives: it is possible to find states (localised around any event) such that all uncertainties ΔX^μ are simultaneously small at wish. Hence an arbitrarily high energy density can be transferred in principle to the geometric background as an effect of localisation alone, which could trap the event in a horizon. In other words, the model is not stable under localisation alone.

A

We recall from [14] the universal regular representation of the usual κ -Minkowski relations

$$[X_{(0)}^0, X_{(0)}^j] = iX_{(0)}^j, \quad [X_{(0)}^j, X_{(0)}^k] = 0. \quad (8)$$

The regular (Weyl) form of the above relations is

$$e^{ih_\mu X_{(0)}^\mu} e^{ik_\mu X_{(0)}^\mu} = e^{i\phi_\mu(h,k)X_{(0)}^\mu}, \quad (9)$$

where

$$\phi^0(h, k) = h^0 + k^0, \quad (10a)$$

$$\vec{\phi}(h, k) = -w(h^0 + k^0, h^0)e^{ik_0}\vec{h} - w(h^0 + k^0, k^0)\vec{k}, \quad (10b)$$

and $w(s, t) = \frac{s(e^t - 1)}{t(e^s - 1)}$ [15, 14]. Above, v, h, k are 4-vectors, and we use the decomposition $v = (v^0, \vec{v})$.

Consider the measure

$$d\mu(\vec{c}, s) = (\delta(|\vec{c}| - 1) + \delta(|\vec{c}|))d\vec{c} ds$$

on $\mathbb{R}^3 \times \mathbb{R}$, where $d\vec{c}, ds$ are the Lebesgue measures on \mathbb{R}^3, \mathbb{R} , respectively; its support is $(S^2 \cup \{0\}) \times \mathbb{R}$. On the space $L^2(\mathbb{R}^3 \times \mathbb{R}, d\mu(\vec{c}, s))$, the operators

$$(X_{(0)}^0 \xi)(\vec{c}, s) = -i \frac{\partial \xi}{\partial s}(\vec{c}, s), \quad (11a)$$

$$(X_{(0)}^j \xi)(\vec{c}, s) = c_j e^{-s} \xi(\vec{c}, s) \quad (11b)$$

fulfil (9). Moreover, they contain precisely one representative for every equivalence class of irreducible representation of the above relations.

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